

# Is Non-Renormalizability a Technical or a Conceptual Problem?

## A Clue from Quasi-Hermitian Representations

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### Abstract

We introduce a new non-Hermitian quantum field Hamiltonian model and prove that it has a real spectrum. The proof goes through the calculation of the closed form positive definite metric operator. We assert that, up to the best of our knowledge, a previous calculations of a closed form metric operator has never been obtained for any non-Hermitian quantum field model. Rather than these new results, we realized that the obtained equivalent Hermitian and the introduced non-Hermitian representations have coupling constants of different mass dimensions which offers a serious suggestion that non-Renormalizability may not be a genuine problem but a technical one. Moreover, the Hermitian form is the well known  $\phi^6$  model which is very interesting in the critical phenomena and early universe studies while the equivalent non-Hermitian representation is a  $\phi^4$ -like theory in which the calculations are more simpler than in the Hermitian  $\phi^6$  theory.

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Two main reasons prevent any theory from playing a role in the description of matter interactions. In a historical order, the first reason is non-Hermiticity of Hamiltonian models which for long time prevented any try to take them into account in the search for a suitable mathematical description that mimic known features of nature. For instance, in the standard model for particle interactions one had to resort to a non-Abelian theory of high group structures to obtain the asymptotic freedom property (**QCD**). Recently, an idea back to Symansic has been stressed and one can show that a simple Abelian theory can have the property of asymptotic freedom [1, 2, 3, 4, 5, 6]. Moreover, also in the standard model, the Hermiticity constraint obliged us to employ a spontaneous symmetry breaking algorithm using only Hermitian scalar field theory which led to the famous Hierarchy problem. On the other hand, in Ref. [1] we showed that the behavior of a non-Hermitian scalar field theory at high energy scales is secure rather than the Hermitian scalar field theory for which the mass parameter and all the dimensionfull parameters blow up leading to the Hierarchy and the cosmological constant problems [12].

The renormalizability of a theory is the second reason that prevents a theory from playing a role in describing a physical system. In this respect, the Particle Physics community celebrated the discovery of the Weinberg-Salam model for its renormalizability and was ready to replace the non-Renormalizable Fermi model introduced to describe Weak interactions. However, another famous problem is still existing regarding the unification of the four forces which up till now is far from reaching a suitable treatment due to the non-Renormalizability of the theory describing gravitational interactions. Besides, in the early universe studies, one need to resort to a theory which is capable of making a strong first order phase transition to account for matter-antimatter asymmetry in the universe [7]. A well known theory that can have such feature is the non-renormalizable  $\phi^6$  scalar field theory [8].

The first reason mentioned above is no longer a holy believe [6, 9] and one can show that there exists an infinite number of Hamiltonians which are neither Hermitian nor  $PT$ -symmetric and have real spectra as well [10]. On the other hand, normalizability have not been stressed in the sense that there are no known technique by which one can get rid of it. In view of the amazing features found in the quasi-Hermitian theories one may wonder if they can offer a clue by which we decide if non-renormalizability of a theory is a technical or a conceptual problem. If it is technical then what is the calculational algorithm that can be used to get rid of it?. In this work we try to answer this question via the introduction

of a new non-Hermitian model and prove that it has a real spectrum. What is amazing in the non-Hermitian quantum field model introduced in this work is that the couplings of the non-Hermitian quantum field model and the equivalent Hermitian quantum field model have different mass dimensions which suggest that normalizability of a theory is representation dependent and one can go, in principle, to an equivalent representation for which the problem does not exist. To elaborate this point, let us talk about the meaning of normalizability. A normalizable theory is that one which has a finite number of infinite amplitudes which can be cured by introducing a finite number of counter terms. On the other hand, a non-renormalizable theory has an infinite number of infinite amplitudes and to turn them finite one has to introduce an infinite number of counter terms and thus one can not execute the calculations because of the existing of infinite number of unknowns (counter terms ). Remarkably, the renormalizability of the theory is determined by the mass dimensions of the coupling in the interaction Hamiltonian and the lowest mass dimension for a theory to be renormalizable is zero. If the coupling has a negative mass dimension then the theory is non-renormalizable. In this work, although we did not know yet a representation in which the non-Renormalizable  $\phi^6$  theory under consideration can be turned renormalizable, we want to spread the message that the dimension (in terms of mass) of the coupling constant is not the same in different equivalent representations for some theory which give us a hope that one can, in principle, find an equivalent representation for a non-renormalizable theory for which the theory in this representation is renormalizable.

In Ref. [10], we introduced the non-Hermitian Hamiltonian of the form;

$$H = p^2 + ig \frac{\{x^3, p\}}{2} \quad (1)$$

where  $p$  is the momentum operator. We showed that this Hamiltonian has the Hermitian equivalent form;

$$h = p^2 + \frac{1}{4}g^2 x^6 \quad (2)$$

One can simply realize that the coupling in the non-Hermitian form is  $g$  while it is  $g^2$  in the equivalent Hermitian form. This realization is not of valuable application in the quantum mechanical level. However, if one is able to map it to the quantum field level of studies it is then a proof that the degree of renormalizability of a theory changes from representation to another.

Now consider the Hamiltonian model of the form

$$\begin{aligned}
H &= \int dx^3 \left( \frac{1}{2} (\nabla \phi(x))^2 + \frac{\pi^2(x)}{2} + \frac{m^2}{2} \phi^2(x) \right. \\
&\quad \left. + \frac{\lambda}{4!} \phi^4(x) + \frac{i\xi}{\sqrt{6!}} \{ \phi^3(x), \pi(x) \} \right), \\
&= H_0 + \epsilon H_I, \\
H_0 &= \int dx^3 \left( \frac{1}{2} (\nabla \phi(x))^2 + \frac{\pi^2(x)}{2} + \frac{m^2}{2} \phi^2(x) \right), \\
H_I &= \int dx^3 \left( \frac{\lambda}{4!} \phi^4(x) + \frac{i\xi}{\sqrt{6!}} \{ \phi^3(x), \pi(x) \} \right).
\end{aligned} \tag{3}$$

where  $\phi$  is a one component scalar field,  $\pi = \dot{\phi} = \partial\phi/\partial t$  and  $\{A, B\}$  is the anticommutator of two operators  $A$  and  $B$ . Also,  $\epsilon$  is a perturbation parameter which can be sent to one at the end. Note that  $\phi$  and  $\pi$  satisfying the commutation relations  $[\phi(x), \pi(y)] = i\delta^3(x-y)$  and  $[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0$ .

The Hamiltonian model in Eq.(3) is non-Hermitian but according to Mostafazadeh [9, 11], if there exists a positive definite metric operator  $\eta$  such that  $\eta H \eta^{-1} = H^\dagger$ , then the spectrum of  $H$  is real. Note also that, if  $\eta$  exists then there exists an equivalent Hermitian Hamiltonian operator  $h$  such that

$$h = \rho H \rho^{-1}, \tag{4}$$

where  $\rho = \sqrt{\eta}$ .

Now from the relation  $\eta H \eta^{-1} = H^\dagger$  and using the form  $\eta = \exp(-Q)$ , one gets

$$H^\dagger = \left( \begin{aligned} &\exp(-Q) H \exp(Q) = H + [-Q, H] \\ &+ \frac{1}{2!} [-Q, [-Q, H]] + \frac{1}{3!} [-Q, [-Q, [-Q, H]]] + \dots \end{aligned} \right)$$

Explicitly we have;

$$\begin{aligned}
\exp(-Q) H \exp(Q) &= H_0 + \epsilon H_I + [-Q, H_0] + [-Q, \epsilon H_I] + [-Q, [-Q, H_0]] + \\
&\quad [-Q, [-Q, \epsilon H_I]] + [-Q, [-Q, [-Q, H_0]]] + [-Q, [-Q, [-Q, \epsilon H_I]]] \dots \\
&= H_0 + \epsilon H_I^\dagger,
\end{aligned}$$

Also we might expand  $Q$  as

$$Q = Q_0 + \epsilon Q_1 + \epsilon^2 Q_2 + \epsilon^3 Q_3 + \dots$$

and thus

$$\begin{aligned}
0 &= [-Q_0, H_0] \Rightarrow Q_0 = 0 \text{ is a good choice.} \\
H_n &= \frac{1}{2}[Q_1, H_0] \\
0 &= \frac{1}{2}[-Q_2, H_0] + \frac{1}{2}[-Q_1, H_I] + \frac{1}{3!}[Q_1, [Q_1, H_0]] \\
0 &= \frac{1}{2}[-Q_3, H_0] + \frac{1}{2}[-Q_2, H_I] + \frac{1}{3!}[Q_2, [Q_1, H_0]] \\
&\quad + \frac{1}{3!}[Q_1, [Q_2, H_0]] + \frac{1}{4!}[-Q_1, [-Q_1, [-Q_1, H_0]]] \\
&\quad + \frac{1}{3!}[-Q_1, [-Q_1, H_I]] \\
0 &= \frac{1}{2}[-Q_4, H_0] + \frac{1}{4}[-Q_3, H_I] + \frac{1}{3!}[-Q_2, [-Q_2, H_0]] \\
&\quad + \frac{1}{5!}[Q_1, [Q_1, [Q_1, [Q_1, H_0]]]] + \frac{1}{3!}[-Q_2, [-Q_1, H_I]] \\
&\quad + \frac{1}{3!}[-Q_1, [-Q_2, H_I]] + \frac{1}{4!}[-Q_1, [-Q_1, [-Q_1, H_I]]] \\
&\quad + \frac{1}{8 \times 4!}[-Q_1, [-Q_1, [-Q_2, H_0]]] \\
&\quad + [-Q_1, [-Q_2, [-Q_1, H_0]]] + [-Q_2, [-Q_1, [-Q_1, H_0]]].
\end{aligned} \tag{5}$$

where  $H_n$  is the non-Hermitian term in the interaction Hamiltonian  $H_I$ . To get a Hermitian representation for the model  $H = H_0 + \epsilon H_I$ , one search for transformations which are able to kill the non-Hermitian interaction term  $H_n$ . In fact, assuming that  $Q(\phi)$  is a real functional of  $\phi$  can do the job because then the transformation of  $H_0$  with a suitable choice of  $Q(\phi)$  will result in another functional of  $\phi$  times  $\pi$ . Considering this, one can expect a great simplification to the above set of coupled operator equations. To show this, consider the transformation of  $H_I$ ;

$$\exp(-Q) H_I \exp(Q),$$

since  $H_I$  is linear in  $\pi$  then the commutators  $[Q_n, H_I]$  are all functionals in  $\phi$  only. Accordingly, the above set has a miraculous simplification such that

$$\begin{aligned}
Q_0 &= 0, \\
H_n &= \frac{1}{2}[Q_1, H_0], \\
Q_2 &= Q_2 = Q_3 \dots = 0.
\end{aligned} \tag{6}$$

Now,  $Q_1$  is an operator when commuted with  $H_0(x)$  gives  $\frac{2i\xi}{\sqrt{6}!} \int dx^3 \{\phi^3(x), \pi(x)\}$ , then one can expect  $Q_1$  to take the form

$$Q_1 = \frac{\xi}{\sqrt{6}!} \int dy^3 \phi^4(y). \quad (7)$$

Thus,

$$\eta = \exp \left( -\xi \int dy^3 \phi^4(y) \right). \quad (8)$$

To obtain the equivalent Hermitian Hamiltonian consider

$$\rho = \exp \left( - \int dy^3 \frac{\xi \phi^4(y)}{2\sqrt{6}!} \right) = \exp \left( - \int dy^3 \omega \right), \quad (9)$$

where we put  $\omega(y) = \frac{\xi \phi^4(y)}{2\sqrt{6}!}$ .

Now

$$\rho H(x) \rho^{-1} = H(x) + \int dy^3 [-\omega(y), H(x)] + \frac{1}{2} \int dz^3 \int dy^3 [-\omega(z), [-\omega(y), H(x)]] + \dots$$

Since only terms dependent on  $\pi$  will be effective in calculating the commutators we have

$$\begin{aligned} [-\omega(y), \frac{\pi^2(x)}{2}] &= [-\omega(y), \pi(x)] \frac{\pi(x)}{2} + \frac{\pi(x)}{2} [-\omega(y), \pi(x)] \\ &= -i \frac{\partial \omega(y)}{\partial \phi(y)} \frac{\pi(x)}{2} \delta^3(x-y) - i \frac{\pi(x)}{2} \frac{\partial \omega(y)}{\partial \phi} \delta^3(x-y) \\ &= -i \frac{\xi}{\sqrt{6}!} \{\phi^3(y), \pi(x)\} \delta^3(x-y) \end{aligned} \quad (10)$$

$$\begin{aligned} \left[ -\omega(z), [-\omega(y), \frac{\pi^2(x)}{2}] \right] &= -i \frac{\xi}{\sqrt{6}!} \delta^3(x-y) [-\omega(z), \{\phi^3(y), \pi(x)\}] \\ &= 2i \left( \frac{\xi}{\sqrt{6}!} \phi^3(y) \right) i \frac{\partial \omega(z)}{\partial \phi(z)} \delta^3(x-z) \delta^3(x-y) \\ &= -2 \left( \frac{\xi}{\sqrt{6}!} \phi^3(y) \right) \left( \frac{\xi}{\sqrt{6}!} \phi^3(z) \right) \delta^3(x-y) \delta^3(x-z) \end{aligned} \quad (11)$$

Then

$$\begin{aligned} \int d^3x \left( \rho \frac{\pi^2(x)}{2} \rho^{-1} \right) &= \int d^3x \frac{\pi^2(x)}{2} - i \int d^3x \int d^3y \left\{ \frac{\xi}{\sqrt{6}!} \phi^3(y), \pi(x) \right\} \delta^3(x-y) \\ &\quad - \int d^3x \int d^3z \int d^3y \left( \frac{\xi}{\sqrt{6}!} \phi^3(y) \right) \left( \frac{\xi}{\sqrt{6}!} \phi^3(z) \right) \delta^3(x-y) \delta^3(x-z) \end{aligned} \quad (12)$$

Also

$$\rho H_n \rho^{-1} = H_n + [-\omega, H_n],$$

but

$$\begin{aligned}
[-\omega(y), H_n(x)] &= [-\omega(y), i \frac{\xi}{\sqrt{6}!} \{\phi^3(x), \pi(x)\}] \\
&= 2 \left( i \frac{\xi}{\sqrt{6}!} \phi^3(x) \right) [-\omega(y), \pi(x)] \\
&= 2 \left( i \frac{\xi}{\sqrt{6}!} \phi^3(x) \right) \left( -i \frac{\partial \omega(y)}{\partial \phi(y)} \right) \delta^3(x-y) \\
&= 2 \left( \frac{\xi}{\sqrt{6}!} \phi^3(x) \right) \left( \frac{\xi}{\sqrt{6}!} \phi^3(y) \right) \delta^3(x-y)
\end{aligned} \tag{13}$$

Collecting the different terms one get the Hermitian Hamiltonian of the form;

$$\begin{aligned}
h &= \int d^3x \left( \frac{1}{2} (\nabla \phi(x))^2 + \frac{\pi^2}{2} + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4 - \left( \frac{\xi}{\sqrt{6}!} \phi^3(x) \right)^2 + 2 \left( \frac{\xi}{\sqrt{6}!} \phi^3(x) \right)^2 \right) \\
&= \int d^3x \left( \frac{1}{2} (\nabla \phi(x))^2 + \frac{\pi^2}{2} + \frac{m^2}{2} \phi^2 + \frac{g}{4!} \phi^4 + \left( \frac{\xi}{\sqrt{6}!} \phi^3(x) \right)^2 \right) \\
&= \int d^3x \left( \frac{1}{2} (\nabla \phi(x))^2 + \frac{\pi^2(x)}{2} + \frac{1}{2} m^2 \phi^2 + \frac{g}{4!} \phi^4 + \frac{\xi^2}{6!} \phi^6 \right).
\end{aligned} \tag{14}$$

This form of the equivalent Hermitian Hamiltonian assures the real spectrum to the non-Hermitian Hamiltonian model in Eq.(3). We assert that this is the first time for a closed form metric operator for a non-trivial quantum field theory to appear in the literature. Remarkably, the most interesting realization in this work comes from the comparison of the mass dimension of the couplings of the interaction Hamiltonians in both of the equivalent models in Eqs.(3,14). In Eq.(3), the coupling of the non-Hermitian interaction Hamiltonian  $\xi$  is  $M^{-1}$  in  $3+1$  dimensions, where  $M$  has a mass dimension. On the other Hand, the Hermitian Hamiltonian in Eq.(14), the  $\phi^6$  term has a coupling  $\propto \xi^2$  which has a mass dimensions of  $-2$  in  $3+1$  dimensions. This is a very interesting result because the dimensionality of the coupling constant determines whether the theory is super-renormalizable (coupling of positive mass dimension), renormalizable (dimensionless coupling) and non-renormalizable (coupling of negative mass dimension). However, our result showed that the coupling dimension of some theory is representation dependent and one can legally aim to find a representation for a non-renormalizable theory in which the theory is renormalizable. This kind of future research is very interesting toward the unification of the four forces in our universe as gravity has a coupling of negative 2 mass dimension. Moreover, the  $\phi^6$  theory which has a coupling of negative 2 mass dimension too seems to be the only candidate to play a role in early universe studies to a count for a strong first order phase

transition needed for the matter-antimatter asymmetry in our universe [8]. In this respect, though in our calculations we were not able to get rid of the non-renormalizability problem however, the status is getting better. Note that, in our work both the non-Hermitian form in Eq.(3) and the Hermitian form in Eq.(14) are non-renormalizable theories. Indeed, the dimension of the coupling constant in the non-Hermitian form moved one step toward the state of renormalizability ( negative one mass dimension instead of two in the Hermitian form). Relying on this result, we want to spread the message that the dimension of the coupling constant for some theory is representation dependent.

The non-Hermitian form introduced in this work has another advantage. In fact, the  $\phi^6$  theory is a very interesting quantum field model for critical phenomena studies and also for the search for models that have bound states. However, the quantum field calculations for this model is not easy as each vertex has an emergence of six lines and thus has a lengthy type of calculations. On the other hand, the equivalent non-Hermitian form in Eq.(3) is a  $\phi^4$ -like theory with only four lines emerging at each vertex and thus the calculations in this representation goes more easily than in the Hermitian representation.

To conclude, we introduced a non-Hermitian quantum field model for which the exact positive definite metric operator has been obtained. It has been shown that the equivalent Hermitian Hamiltonian is the famous  $\phi^6$  scalar field theory. Rather than these novel discoveries, the work introduced here has two main interesting features. First, the coupling constant of both the equivalent Hermitian and non-Hermitian forms has different mass dimensions which gives a clear message that the degree of renormalizability of a theory is representation dependent. Accordingly, one may hope to get a renormalizable representation for a non-renormalizable one. Second, the non-Hermitian form of the  $\phi^6$  theory is a  $\phi^4$ -like theory which means that the calculations in the non-Hermitian representation are more simpler than in the Hermitian one. In fact, the  $\phi^6$  theory with positive couplings is in the same class of universality with the  $\phi^4$  theory which in turn is non-perturbative near the critical point. However, working with the  $\phi^6$  theory can give second order phase transition in first order calculation a result which can not gotten by the  $\phi^4$  theory. On the other hand, the calculations are more harder in the  $\phi^6$  theory but the non-Hermitian form has the same simplicity as in the case of the  $\phi^4$  model. Accordingly, in the critical phenomena studies, it is better to work with the non-Hermitian form of the  $\phi^6$  theory than either in the  $\phi^4$  or  $\phi^6$



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